Unimodality and Chain Decompositions

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University of Minnesota Student Combinatorics Seminar

September 24, 2020

Sequence $a_0, a_1, \ldots, a_n$ of real numbers is *symmetric* if, for all k,

$$a_k = a_{n-k}.$$

Proposition

Given n, the following binomial coefficient sequence is symmetric

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$

Proof.

To see this algebraically, note that

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

For a combinatorial proof, let $[n] = \{1, \ldots, n\}$ and define

$$\binom{[n]}{k} = \{S \mid S \subseteq [n],\ \#S = k\}.$$

Then $f : \binom{[n]}{k} \to \binom{[n]}{n-k}$ where $f(S) = [n] - S$ is a bijection. $\square$

Sequence $a_0, a_1, \ldots, a_n$ is *unimodal* if there is an index m with

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \geq a_n.$$

Unimodal squences abound in combinatorics, algebra, and geometry; see the survey articles of Stanley, Brenti, and Brändén.

Proposition

Given n, the following binomial coefficient sequence is unimodal

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$

Proof.

For an algebraic proof, since the sequence is symmetric it suffices to prove that $\binom{n}{k} \leq \binom{n}{k+1}$ for $k < n/2$. This is equivalent to

$$\frac{n!}{k!(n-k)!} \leq \frac{n!}{(k+1)!(n-k-1)!} \iff k+1 \leq n-k.$$

which is iff $2k+1 \leq n \iff k < n/2$.
A combinatorial proof can be given by using a lattice path method called the Reflection Principle (Sagan). $\square$

We will give a combinatorial proof of the previous results using chain decompositions. Let $(P, \trianglelefteq)$ be a finite poset (partially ordered set). If $x, y \in P$ then a *saturated x–y chain* is

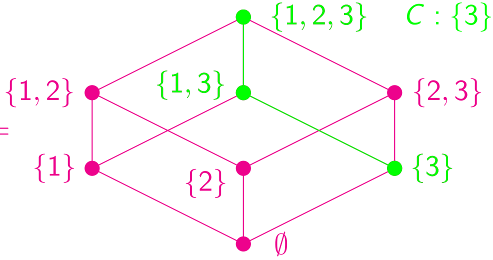
$$C : x = x_0 \lhd x_1 \lhd \ldots \lhd x_m = y$$

where each $\lhd$ is a cover. We assume P is *ranked* meaning

1. P has a unique minimum element $\hat{0}$,
2. if $x \in P$, the lengths of all saturated $\hat{0}$–x chains are equal.

Let $\operatorname{rk} x$ be the common chain length and $\operatorname{rk} P = \max_{x \in P} \operatorname{rk} x$.

Ex. Consider the *Boolean algebra B_n* of all subsets $S \subseteq [n]$ ordered by inclusion. Then B_n is ranked with $\operatorname{rk} S = \#S$ and $\operatorname{rk} B_n = n$.

$B_3 =$ 

$C : \{3\} \lhd \{1,3\} \lhd \{1,2,3\}$

Let $r_k(P)$ be the number of elements at rank k in P with $\mathrm{rk}\, P = n$. P is *rank symmetric/unimodal* if the sequence $r_0(P), \ldots, r_n(P)$ is. The *center* of a saturated x–y chain in a ranked poset P is

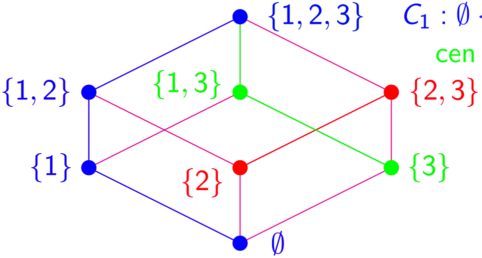
$$\mathrm{cen}\, C = \frac{\mathrm{rk}\, x + \mathrm{rk}\, y}{2}.$$

A *chain decomposition (CD)* of P is a partition of P into disjoint, saturated chains $P = \uplus_i C_i$. A *symmetric chain decomposition (SCD)* is a CD with $\mathrm{cen}\, C_i = n/2$ for all i.

Theorem

If P has a SCD then it is rank symmetric and rank unimodal. $\square$

Ex. $r_0(B_3), \ldots, r_3(B_3) = 1, 3, 3, 1$ symmetric and unimodal.



$C_1 : \emptyset \lhd \{1\} \lhd \{1,2\} \lhd \{1,2,3\}$

$\mathrm{cen}\, C = \frac{1+3}{2} = 2.$

$C_2 : \{2\} \lhd \{2,3\}$

$C_3 : \{3\} \lhd \{1,3\}$

How do we find an SCD of B_n? Associate with each $S \subseteq [n]$ a binary word $w = w_S = w_1 \ldots w_n$ where

$$w_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Form the *Greene-Kleitman core of w*, $\mathrm{GK}(w)$, by pairing any $w_i = 0$ and $w_{i+1} = 1$, then pairing any 0 and 1 separated only by already paired elements, etc. Any unpaired w_j is called *free* and the free elements of w must be a sequence of ones followed by a sequence of zeros. Given core κ, form a chain C_κ by starting with the word which is zero outside κ and then turning the free zeros to ones from left to right.

Theorem (Greene-Kleitman)

The C_κ as κ varies over all possible cores form an SCD of B_n. □

Ex. If $S = \{1, 5, 7, 8\} \subset [9]$ then $w = w_S = 100010110$.

$$\kappa = \mathrm{GK}(w) = **\widehat{0\widehat{01}\widehat{01}1}*\,.$$

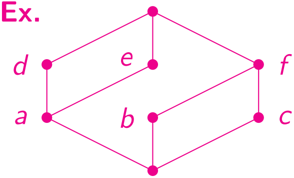
$$C_\kappa : 000010110 \lhd 100010110 \lhd 110010110 \lhd 110010111.$$

The Sperner property. An *antichain* in a poset P is a set A of elements which are pairwise incomparable. If P is ranked, then the elements at a given rank are an antichain. So if $a(P)$ is the size of a largest antichain of P then

$$a(P) \geq \max_k r_k(P). \tag{1}$$

It is possible for this inequality to be strict.

Ex.



$a(P) = 4$ because of $A = \{b, c, d, e\}$.

The maximum rank size is 3.

Call P *Sperner* if (1) is an equality.

Theorem

If P has and SCD then it is Sperner. $\square$

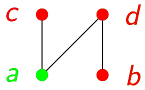
There is a more general notion of *strongly Sperner* where one looks at subposets of P whose longest chain has length ℓ for all possible ℓ. The previous theoren still holds for strongly Sperner.

Distributive lattices. A *lattice*, L, is a poset such that every $x, y \in L$ have a greatest lower bound (meet), $x \wedge y$, and a least upper bound (join), $x \vee y$. Call L *distributive* if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A *(lower order) ideal* of a poset P is $I \subseteq P$ such that

$$y \in I \quad \text{and} \quad x \trianglelefteq y \implies x \in I.$$

$P =$ 

$\{a, b, d\}$ is an ideal

$\{b, c, d\}$ is not an ideal

For P a finite poset, let $L(P)$ be all ideals of P ordered by inclusion.

Theorem (Fundamental Thm. of Finite Distributive Lattices)

P a finite poset implies $L(P)$ is a distributive lattice. And any finite distributive lattice is isomorphic to $L(P)$ for some poset P. $\square$

Open Problem: Characterize distributive lattices having SCDs.

Ex. B_n is a lattice with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$. Also, B_n is distributive since $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$. If A_n is an n-element antichain then $B_n \cong L(A_n)$.

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THANKS FOR LISTENING!